International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Quantum Kinematic Theory of the Poincaré Group in Two-Dimensional Spacetime

J. Krause[1]

Received June 19, 1997

Non-Abelian quantum kinematics is applied to the Poincaré group $\mathcal{P}_+^\uparrow(1, 1)$, as an example of the *quantization-through-the-symmetry* approach to quantum mechanics. Upon *quantizing* the group, generalized Heisenberg commutation relations are obtained, and a closed Heisenberg–Weyl algebra follows. Then, according to the general theory, the three basic quantum-kinematic invariant operators are calculated; these afford the *superselection* rules for diagonalizing the incoherent rigged Hilbert space $\mathcal{H}(\mathcal{P}_+^\uparrow)$ of the regular representation. This paper examines only one of these diagonalization schemes, while introducing a *irreducible spacetime representation* carried by *isotopic plane-wave* eigenvectors of two compatible superselection operators (which define a Poincaré-invariant linear 2-momentum). Thereafter, the principle of microcausality produces massive *2-spinor* isotopic states in 1 + 1 Minkowski space. The Dirac equation is thus *deduced* within the quantum kinematic formalism, and the familiar Jordan–Pauli propagation kernel in 2-dimensional spacetime is also obtained as a Hurwitz *invariant integral* over the group manifold. The main interest of this approach lies in the adopted *group-quantization* technique, which is a strictly deductive method and uses exclusively the assumed Poincaré symmetry.

1. INTRODUCTION

This paper deals with some irreducible representations of the Poincaré group $\mathcal{P}_+^\uparrow(1, 1)$. Since this matter settles a rather old issue, on first sight the subject of this paper may well seem obsolete (Wigner, 1939; Bargmann and Wigner, 1946). However, relativistic symmetry will be considered here under a new (somehow heterodox) quantum perspective, in which we shall tighten the twining of quantum mechanics with the theory of special relativity. Quite generally, through the chosen example, we hope to convince the reader that there is still more to be said about the role of symmetry in quantum physics.

[1] Facultad de Física, Pointificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile; e-mail: jkrause@chopin.fis.puc.cl.

0020-7748/98/0300-0759$15.00/0

The present work addresses the problem of *quantizing the group* of inhomogeneous proper Lorentz transformations in 2-dimensional Minkowski spacetime. With no pretension toward rigor, suffice it to say that by 'quantizing a Lie group' we mean that one associates with the parameter q^a of the group a complete set of commuting Hermitian operators Q^a (i.e., $q^a \to Q^a$), which behave as *generalized position operators* on the group manifold, for they have the parameters for spectra. This is the starting idea of *non-Abelian quantum kinematics*, as it has been introduced in the literature over the recent years. A general discussion of the issues involved in this formalism is given in Krause (1994a); a recent *heuristic* interpretation of the quantum kinematic approach to dynamics can be found in Krause (1997a). [To avoid confusion, let us here remark that *Lie group quantization* is a mathematical formalism bearing no relation whatever with the formalism of so-called 'quantum groups' now in vogue. See, for instance, Pillin (1994). As is well known, 'quantum groups' are not *groups*. They are '*q*-deformed Lie algebras', becoming Hopf algebras. For a very lucid exposition of the subject of 'quantum groups' see Biedenharn and Lohe (1995).]

The group-theoretic *position operators* and the *non-Abelian momentum operators* (afforded by the *generators* of the regular representation) satisfy well-defined *generalized Heisenberg commutation relations* (Krause, 1985). As we have shown in our previous work, together with the non-Abelian Lie algebra and the adjoint representation of the group, the new kinematic commutation relations lead to a closed *generalized Heisenberg–Weyl algebra* (Krause, 1991), as well as to a *generalized enveloping algebra* (Krause, 1993a) associated with the quantized group. These new structures are *generalizations* of the old structure indeed, for they extend to *non-Abelian Lie groups* the traditional Heisenberg–Weyl structure stemming from the *Abelian group* of rigid space transport in a Cartesian scaffolding, on which *canonical quantization* has been resting hitherto (Weyl, 1931; Komar, 1971). As a matter of fact, in non-Abelian quantum kinematic theory, *quantization* becomes a precise, consistent, systematic, and general group-theoretic procedure, able to yield new kinematic foundations for quantum dynamics (Krause, 1994a). In this sense, we claim that the old notion of 'quantization' should lose its 'philosophical flavor' as historically related with a kind of mysterious, metaphysical connection between quantum theory and some thoroughly ad hoc classical analog theories (Jammer, 1989), for which the Ehrenfest theorem is too narrow an argument for the old principle of correspondence. In fact, from a strictly physical point of view, one has to admit that most relevant quantum systems known today have no classical analog at all.

Moreover, the actual interest of a geometric generalization of the notion of 'quantization' is *not* purely philosophical or academic. Indeed, it has been found that *quantization* of an *r*-dimensional Lie group produces a set of *r* *basic quantum kinematic invariant operators.* These are functions of the generators (*and* of the position operators) that commute with all the generators; they may be calculated in a standard manner (Krause, 1991) and, furthermore, *all* the invariants of the group (including the traditional ones, such as the Casimir operators, for instance) are functions thereof (Krause, 1993a). Hence, the kinematic invariant operators yield the several *superselection rule* schemes of the theory, by means of which physically meaningful *irreducible representations* can be introduced. Very especially, such irreducible representations can be defined in configuration spacetime (as a homogeneous space for the action of the group), in terms of wave functions that satisfy a set of *invariant wave equations* (containing the Schrödinger equation, if any), the form of which one deduces quite generally from the chosen superselection rules (Krause, 1994a). In this manner, the most striking achievement of non-Abelian quantum kinematics is that this theory produces the *propagation kernel* of a system as the transition amplitudes between such irreducible configuration states, which are given by a *Hurwitz invariant integral* defined over the group manifold. To sum up, the *irreducible configuration representations* describe a quantum model of a system that remains invariant under the action of the group. These are the main features which enhance the *quantized group* with a rich mathematical structure, and offer huge possibilities of new applications for Lie groups in quantum theory.

For a successful application of the main formal structure of quantum kinematic theory, see Krause (1996), where the Schrödinger equation and the Feynman propagation kernel are *deduced* in a strict group-theoretic manner, for the Landau group characterizing a point charge moving in a constant magnetic field. [The concept of the complete *symmetry group* of a mechanical system plays here a central role. This concept has been introduced recently in the literature, for the classical Kepler problem (Krause, 1994b).] Early successful models of group quantization can be found in Krause (1986, 1988) for the simple harmonic oscillator and for a Galilean free particle, respectively. In those years, however, the theory of the *quantum kinematic superselection rules* was missing, and the author was not aware of the general formalism of quantum kinematic theory. Nevertheless, neither *canonical quantization* nor *path integral methods* were used in these early references.

It is the intention of this paper to adopt this group-theoretic quantization method in order to initiate a study of quantum kinematic models describing Poincaré-invariant elementary systems in a two-dimensional world. Let us here underline the introductory character of this article, since the endeavor developed in the sequel is by no means complete. Here we touch only on some basic kinematic features of the resulting *toy models* of 2-dimensional massive free fermions. Bosons, internal structure, and possible dynamics for such systems, as predicted by the quantum kinematic analysis of $\mathcal{P}(1, 1)$, will be considered in forthcoming papers.

The organization of this paper follows the standard formalism of quantum kinematics. In Section 2 we quantize $\mathcal{P}_{+}^{\uparrow}(1, 1)$, obtaining the generalized Heisenberg commutation relations, as well as the closed Heisenberg–Weyl quantum kinematic algebra. (Other useful commutation relations are also considered in this section, for future reference.) Section 3 is a brief study of the relativistic quantum-kinematic invariant operators. Here we discuss the superselection rules, in the light of the heuristic postulates of the theory. Next, in Section 4 we introduce a spacetime representation of the Poincaré group within the rigged Hilbert space that carries the regular representation. Thus we get *isotopic plane wave states* which satisfy a maximal set of two compatible superselection rules, whose 'wave vector' corresponds to an isotopic (i.e., internal) linear 2-momentum yielding a proper mass. (Massless states are *not* considered in this paper.) *Noncausal* propagation kernels are obtained, however, from the transition amplitudes of these basic spacetime vectors. Therefore, in Section 5 we adopt the *principle of microscopic causality,* which brings in *massive 2-spinors*, transforming appropriately under Poincaré transformations in Minkowski spacetime. They obey automatically (that is, by construction) the *Dirac equation* (which is thus *deduced* in the present theory). Moreover, the transition amplitudes of these causal spacetime vectors yield precisely the *Jordan–Pauli propagator* function in 1 + 1 dimensions, which is here given by a Hurwitz invariant integral over the group manifold, as expected. Some features of the quantum kinematic theory of the massive free-fermion causal propagator are briefly discussed in this section. Finally, in Section 6 we present our concluding remarks and some perspectives for future work.

This paper includes two appendices: Appendix A revisits some special features of Poincaré transformations in 1 + 1 dimensions. Appendix B is devoted to the regular representation of $\mathcal{P}_{+}^{\uparrow}(1, 1)$. These appendices summarize the required prolegomena for understanding this paper.

2. $\mathcal{P}_{+}^{\uparrow}(1, 1)$ QUANTUM KINEMATICS

We first proceed to *quantize* the Poincaré group in 1 + 1 dimensions, according to the general formalism developed in our previous work. As we have already noted, this leads to generalized Heisenberg commutation relations, which produce the associated Heisenberg–Weyl quantum kinematic algebra.

2.1. Generalized Heisenberg Commutation Relations

The best (if not the only) way of achieving *group quantization* is within the rigged Hilbert space $\tilde{\mathcal{H}}(\mathcal{P}_{+}^{\uparrow})$ that carries the regular representation (cf.

Appendix B). So, let us define *generalized position operators* of the group manifold $M(\mathcal{P}_+^{\uparrow})$ by means of the following spectral integrals:

$$Q^a = \mu_0 \iint dq^0\, dq^1 \int_{-1}^{1} dq^2\, \gamma^2(q^2) |q^0, q^1, q^1\rangle q^a \langle q^0, q^1, q^2| \quad (2.1)$$

for $a = 0, 1, 2$. Here we have used the invariant measure given in equation (B.1), and we have taken into account the resolution of the identity stated in equation (B.3). Therefore, as a consequence of equation (B.4), one has $Q^a|q\rangle = q^a|q\rangle$ and $[Q^a, Q^b] = 0$. Hence, the Q's provide a complete set of commuting Hermitian operators in $\tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$.

It is interesting to consider the active transformation law which brings the position operators Q^a from the 'Schrödinger picture' into the 'Heisenberg picture' of the kinematics; namely, one defines parameter-dependent operators $\hat{Q}^a(q) = U^{\dagger}(q)Q^aU(q)$, where $U(q)$ denotes the unitary operators of the representation (Krause, 1985). In this fashion, one notes that the generalized position operators transform *covariantly* upon the group law. In fact, according to equations (A.3) and (2.1), we obtain the 'left' transformation law:

$$\begin{aligned} U_L^{\dagger}(q)Q^0(U_L(q) &= q^0 + \gamma(q^2)(Q^0 - q^2Q^1) \\ U_L^{\dagger}(q)Q^1U_L(q) &= q^1 + \gamma(q^2)(Q^1 - q^2Q^0) \\ U_L^{\dagger}(q)Q^2U_L(q) &= (q^2 + Q^2)(1 + q^2Q^2)^{-1} \end{aligned} \quad (2.2)$$

as well as the 'right' transformation law, which reads

$$\begin{aligned} U_R^{\dagger}(q)Q^0U_R(q) &= Q^0 + \gamma(Q^2)(q^0 - Q^2q^1) \\ U_R^{\dagger}(q)Q^1U_R(q) &= Q^1 + \gamma(Q^2)(q^1 - Q^2q^0) \\ U_R^{\dagger}(q)Q^2U_R(q) &= (Q^2 + q^2)(I + Q^2q^2)^{-1} \end{aligned} \quad (2.3)$$

We then evaluate these transformations in a small neighborhood of the identity, and thus we obtain *generalized Heisenberg commutation relations* obeyed by the position operators and the *non-Abelian* generators of the regular representation. In the present case, one gets [cf. equation (A.5)]

$$\begin{array}{lll} [Q^0, L_0] = i\hbar, & [Q^1, L_0] = 0, & [Q^2, L_0] = 0 \\ [Q^0, L_1] = 0, & [Q^1, L_1] = i\hbar, & [Q^2, L_1] = 0 \\ [Q^1, L_2] = -i\hbar Q^1, & [Q^1, L_2] = -i\hbar Q^0, & [Q^2, L_2] = i\hbar\gamma^{-2}(Q^2) \end{array} \quad (2.4a)$$

and

$$[Q^0, R_0] = i\hbar\gamma(Q^2), \quad [Q^1, R_0] = -i\hbar\gamma(Q^2)Q^2, \quad [Q^2, R_0] = 0$$
$$[Q^2, R_1] = -i\hbar\gamma(Q^2)Q^2, \quad [Q^1, R_1] = i\hbar\gamma(Q^2), \quad [Q^2, R_1] = 0$$
$$[Q^0, R_2] = 0, \quad [Q^1, R_2] = 0, \quad [Q^2, R_2] = i\hbar\gamma^{-2}(Q^2) \tag{2.4b}$$

This is the point where the quantum principle begins to emerge from the *classical* theory of $\mathcal{P}^{\uparrow}_{+}(1, 1)$. Notice that these commutation relations are not all canonical (neither do they close to form a finite algebra).

Let us here remark that the mathematical importance of quantum kinematics stems from the fact that the sets $\{Q^0, Q^1, Q^2; L_0, L_1, L_2\}$ and $[Q^0, Q^1, Q^2; R_0, R_1, R_2\}$ [and not just the set of generators $\{L_0, L_1, L_2\}$, nor $\{R_0, R_1, R_2\}$] are the *irreducible sets* of Hermitian operators that characterize the carrier Hilbert space $\tilde{\mathcal{H}}(\mathcal{P}^{\uparrow}_{+})$ defined by the group structure of $\mathcal{P}^{\uparrow}_{+}(1, 1)$ itself. [For this notion, see for instance, Fonda and Ghirardi (1970).]

In this manner, we have arrived at one of the most important results of non-Abelian Lie group quantization, which points beyond the canonical quantization formalism. Certainly, such generalized commutation relations are of potential value for physics. We next explore some of their consequences.

2.2. Heisenberg–Weyl Quantum Kinematic Algebra

One obtains a *closed quantum kinematic algebra* associated with $\mathcal{P}^{\uparrow}_{+}(1, 1)$, as follows. Using some general properties of the adjoint representation [see Eq. (A.7)] yields the following commutation relations (Krause, 1991):

$$[A^c_b(Q), L_a] = i\hbar f^d_{ab} A^c_d(Q), \qquad [A^c_b(Q), R_a] = i\hbar f^c_{ad} A^d_b(Q) \tag{2.5}$$

One interprets equations (2.5), together with the Lie algebra stated in equations (B.15) and the fact that $[A^b_a(Q), A^d_c(Q)] \equiv 0$, as the *generalized Heisenberg–Weyl algebra* of the quantized group.

The closed commutation relations (2.5) are very helpful in order to analyze the nonclosed Heisenberg commutation relations (2.4). For instance, as miscellaneous examples, in the case of $\mathcal{P}^{\uparrow}_{+}(1, 1)$, equations (2.5) yield

$$[\gamma(Q^2)(Q^0 + Q^2Q^1), R_0] = i\hbar, \qquad [\gamma(Q^2)(Q^1 + Q^2Q^0), R_1] = i\hbar \tag{2.6a}$$

$$[\gamma(Q^2)(Q^1 + Q^2Q^0), R_0] = 0, \qquad [\gamma(Q^2)(Q^0 + Q^2Q^1), R_1] = 0 \tag{2.2b}$$

as well as

$$[\gamma(Q^2)(I \pm Q^2), R_2] = \pm i\hbar\gamma(Q^2)(I \pm Q^2) \tag{2.7}$$

and

$$[\gamma(Q^2)(Q^0 \pm Q^2Q^2), R_2] = \pm i\hbar\gamma(Q^2)(Q^1 \pm Q^2Q^0) \tag{2.8a}$$

$$[\gamma(Q^2)(Q^1 \pm Q^2Q^0), R_2] = \pm i\hbar\gamma(Q^2)(Q^0 \pm Q^2Q^1) \tag{2.8b}$$

(Of course, one can also obtain these commutators directly, by inspection of the generalized Heisenberg commutation relations, but the calculations are not so simple.)

These results are not just minutiae, for they all play important roles in the present theory. Very especially, observe that in this fashion one gets, besides $[Q^\mu, L_\nu] = i\hbar\delta^\mu_\nu$, as appears in equations (2.4a), the following *canonical* commutation relations:

$$[\bar{Q}^\mu, R_\nu] = -i\hbar\delta^\mu_\nu \tag{2.9}$$

which do *not* appear in (2.4b). Here we have defined the operators $\bar{Q}^\mu$ using the inversion law of the parameters, given in equation (A.4), i.e., we define

$$\bar{Q}^0 = -\gamma(Q^2)(Q^0 + Q^2Q^1), \qquad \bar{Q}^1 = -\gamma(Q^2)(Q^1 + Q^2Q^0), \qquad \bar{Q}^2 = -Q^2 \tag{2.10}$$

This means that $\bar{Q}^\mu|q\rangle = \bar{q}^\mu|q\rangle$ and $\bar{Q}^\mu|\bar{q}\rangle = q^\mu|\bar{q}\rangle$ hold. Within the left regular representation, the kinematic law for the *inverse-position operators* reads

$$U_L^\dagger(q)\bar{Q}^\mu U_L(q) = \bar{Q}^\mu + \bar{\Lambda}^\mu{}_\nu(Q^2)\bar{q}^\nu \tag{2.11}$$

As we see, the real interest of the new position operators $\bar{Q}^\mu$ is that they are *Lorentz invariant* Hermitian operators [since, plainly, $q^\mu = 0 \Rightarrow \bar{q}^\mu = 0$]. The following transformation law is also worth noting:

$$U_L^\dagger(q)\gamma(Q^2)(I \pm Q^2)U_L(q) = \gamma(q^2)(1 \pm q^2)\gamma(Q^2)(I \pm Q^2) \tag{2.12}$$

for it will play a role in the sequel.

We have presented these matters here in order to show the formal possibilities offered by the quantum kinematic treatment of the group, and also for future reference.

3. QUANTUM KINEMATIC SUPERSELECTION RULES

Henceforth we adopt the *left* regular representation of $\mathcal{P}^\uparrow_+(1, 1)$ (cf. Appendix B) as the underlying working frame of the theory. We now proceed to study the set of basic quantum kinematic invariant operators of the group, from which the superselection rules of the theory are obtained.

3.1. Quantum Kinematic Invariant Operators

It is well known that the Lie algebra (B.15) associated with $\mathcal{P}^\uparrow_+(1, 1)$ has just *one* Casimir operator, which is given by the scalar operator

$$W = \eta^{\mu\nu} L_\mu L_\nu \equiv \eta^{\mu\nu} R_\mu R_\nu \tag{3.1}$$

where $\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+-)$ is the Minkowski metric. According to the traditional approach to this subject, this operator yields the 2-dimensional Klein–Gordon equation, related with the only superselection rule available in $\tilde{\mathcal{H}}(\mathcal{P}^\uparrow_+)$. However, the realm of quantum kinematics is much broader than the traditional standpoint, for according to this perspective one obtains more superselection rules than in the traditional theory of Lie algebras. In fact, one has a *larger* quantum kinematic Heisenberg–Weyl algebra, and therefore new invariant operators follow as linear combinations of the generators, whose matrix coefficients belong to the *anti-adjoint representation* [given by the inverse matrix of the one defined in equation (A.10)] considered as *functions of the generalized position operators*, as we shall see presently.

As another important application of the quantum kinematic algebra (2.5), let us here recall some features leading to the theory of the *basic quantum kinematic invariants* of the group (Krause, 1991). Within the left regular representation the basic invariants are given by the general formula

$$R_a(Q; L) = R^\dagger_a(Q; L) = \bar{A}^b_a(Q) L_b \tag{3.2}$$

Indeed, $[\bar{A}^b_a(Q), L_b] = i\hbar f^b_{ab} = 0$ follows for $\mathcal{P}^\uparrow_+(1, 1)]$, and, clearly, from equations (B.14), one gets

$$U^\dagger_L(q) R_a(Q; L) U_L(q) = R_a(Q; L) \tag{3.3}$$

Hence, these are the desired invariant operators. In the 'Q-representation' of quantum kinematics, it is an easy matter to obtain $R_a(Q; L)|q\rangle = i\hbar Y_a(q)|q\rangle$, and thus one identifies these operators as the *generators of the right regular representation,* acting as *invariant operators* within the left regular representation of $\mathcal{P}^\uparrow_+(1, 1)$ [cf. equation (B.16)]. Bear in mind that this can be achieved if, *and only if*, one *quantizes* the group.

In the case of $\mathcal{P}^\uparrow_+(1, 1)$, the formalism sketched above becomes rather simple. Substituting from equations (A.7) into equation (3.2) yields the following *three* basic invariants of the Poincaré group in 1 + 1 dimensions:

$$R_0(Q; L) = \gamma(Q^2)(L_0 - Q^2 L_1) \tag{3.4a}$$

$$R_1(Q; L) = \gamma(Q^2)(L_1 - Q^2 L_0) \tag{3.4b}$$

$$R_2(Q; L) = Q^1 L_0 + Q^0 L_1 + L_2 \tag{3.4c}$$

These operators are Hermitian. They are not all compatible, however, for they satisfy the right Lie algebra (B.15a). We see that the right-displacement-generators $R_\mu = R_\mu(Q; L)$ are given by a *quantized* Lorentz transformation of the left-displacement generators L_μ; i.e., equations (3.4a) and (3.4b) read $R_\mu = \Lambda^\nu{}_\mu(Q^2) L_\nu$. On the other hand, as we see from equation (3.4c), the right-boost-generator $R_1 = R_2(Q; L)$ corresponds to a *total pseudo-Euclidean angular momentum operator* related to hyperbolic Lorentz rotations in 2-dimensional flat spacetime. Note that, in this theory, L_μ is the genuine *linear momentum operator* and L_2 is the genuine *hyperbolic angular momentum operator* (cf. below).

3.2. Superselection Rules: Heuristic Postulates

At this point, we are ready to consider the *quantum kinematic superselection rules*. The statement "$\mathcal{P}^{\uparrow}_{+}(1, 1)$ is a *symmetry group* of a physical system" means that this group acts on the manifold of the physical states of the system, transforming one physical state into another. Moreover, it *also* means that the action of the group does not change the dynamical structure of the system. Hence, any reasonable physical interpretation of a quantum kinematic model based on $\mathcal{P}^{\uparrow}_{+}(1, 1)$ must assume the following heuristic postulate:

The allowable physical pure states of a system correspond to simultaneous eigenvectors in $\tilde{\mathcal{H}}(\mathcal{P}^{\uparrow}_{+})$ of a maximal set of compatible quantum kinematic Hermitian invariant operators of the group.

This means that every *maximal* set of compatible invariant operators yields a set of *superselection rules*, by means of which the *incoherent* Hilbert space $\tilde{\mathcal{H}}(\mathcal{P}^{\uparrow}_{+})$ can be diagonalized into invariant Hilbert subspaces, each carrying an irreducible physical representation of the group (namely, a model of the system).

On the other hand, concerning the properties that characterize the dynamical structure of the system, one should also consider the following postulate:

The eigenvalues of the operators pertaining to a maximal set of superselection rules correspond to some intrinsic physical properties characterizing the structure of the system.

The eigenvalues of the quantum kinematic superselection rule operators are the labels characterizing the respective invariant Hilbert spaces which carry the quantum model of a system. The second heuristic postulate enhances some features of the *right* regular representation with the character of an *isotopic* structure, describing some properties of the permanent *internal* nature of the system. [Recall that in these interpretations one assumes the left regular representation as the adopted working frame (Krause, 1994a, 1997a).]

Now, according to these postulates, one can *diagonalize* the quantum kinematic structure of the Poincaré group $\mathcal{P}^{\uparrow}_{+}(1, 1)$ in one of the following schemes: either one uses the compatibility conditions (a) $[R_0, R_1] = 0$, (b) $[W, R_2] = 0$, (c) $[W, R_0] = 0$, or else one uses (d) $[W, R_1] = 0$. Solving for the eigenvalue problems of the corresponding compatible invariant operators reduces the left regular representation into irreducible representations, which one hopes to interpret properly. In this sense, the quantum kinematic theory of $\mathcal{P}_+^{\uparrow}(1, 1)$ is automatically relativistic invariant.

In the sequel we only consider models that arise from the first maximal set of superselection operators $\{R_0, R_1\}$. As we shall see presently, this set produces the theory of the Dirac equation of massive free fermions in 1 + 1 dimensions as a very special consequence. Quantum kinematic models stemming from the remaining superselection rules [i.e., $\{W, R_2\}$, $\{W, R_0\}$, and $\{W, R_1\}$] will be considered elsewhere.

4. SPACETIME REPRESENTATIONS

As follows from the previous discussion, what is still missing in this approach is a *spacetime description* of the quantum kinematic models in terms of wave functions evolving in spacetime according to well-defined wave equations. We shall here attain such a description, in accord with the geometric demands of special relativity.

4.1. Geometric Representations

To this end, we first consider a kind of vector $|x\rangle = |x^0, x^1\rangle \in \tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$ which maintains a one-to-one correspondence with the events $x = (x^0, x^1)$, and is such that, by construction, one has

$$U_L(q^0, q^1, q^2)|x^0, x^1\rangle = |\gamma(q^2)(x^0 - q^2x^1) + q^0, \gamma(q^2)(x^1 - q^2x^0) + q^1\rangle \qquad (4.1)$$

for all q and x. These vectors carry a *geometric representation* of the spacetime realization of $\mathcal{P}_+^{\uparrow}(1, 1)$ given in equations (A.1). In fact, the property stated in (4.1) entails the following transformation law for wave functions $\psi(x) = \langle x|\psi\rangle$ [with $|\psi\rangle \in \tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$] defined on the spacetime arena:

$$\langle x|U_L^{\dagger}(q)|\psi\rangle = \psi[\Lambda^{\mu}{}_{\nu}(q^2)x^{\nu} + q^{\mu}] = \psi_{\tilde{q}}(x) \qquad (4.2)$$

One has enormous freedom for building such geometric representations within $\tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$. Indeed, it can be shown that a necessary and sufficient condition for the vectors $|x\rangle$ to be endowed with property (4.1) is that they have the following general form:

$$|x; \xi\rangle = \int d\mu(q)\, \xi^*[\Lambda^{\mu}{}_{\nu}(\bar{q}^2)(x^{\nu} - q^{\nu})]\,|q\rangle \qquad (4.3)$$

where the *generating wave function* $\xi(x)$ remains arbitrary. Hence, in order to obtain a specific model, $\xi(x)$ must be determined on some physical grounds (Krause, 1994a).

Before tackling this problem, let us also observe that an infinitesimal Poincaré transformation of $|x;\, \xi\rangle$ yields the usual spacetime realization of the left-displacement generators (i.e., 2-momentum operators):

$$L_0 |x;\, \xi\rangle = i\hbar(\partial/\partial x^0)|x;\, \xi\rangle, \qquad L_1 |x;\, \xi\rangle = i\hbar(\partial/\partial x^1)|x;\, \xi\rangle \quad (4.4a)$$

while for the left-Lorentz-boost generator one obtains

$$L_2 |x;\, \xi\rangle = -i\hbar[x^1(\partial/\partial x^0) + x^0(\partial/\partial x^1)]|x;\, \xi\rangle \quad (4.4b)$$

We do not consider central ray extensions [by $U(1)$] of the $1 + 1$ Poincaré group in this paper. This subject will be studied elsewhere. The extension of the formalism of non-Abelian quantum kinematics, from 'true' (i.e., vector) representations to 'projective' ray representations, faces no difficulties (Krause, 1994a). Irreducible projective representations of $\mathcal{P}_+^\uparrow(1, 1)$ have been found recently by Bose (1996) as an application of the Kirillov theory.

4.2. Isotopic Linear Momenta

Bearing in mind the basic postulates of the theory, let us examine the superselection rules arising from the *right*-displacement generators. That is, we require very specific *spacetime kets* which may be realizable states of the system. So, we look for a special generating function $\xi(x;\, \rho)$ that satisfies the superselection rules:

$$R_0 |x;\, \rho;\, \xi\rangle = \rho_0 |x;\, \rho;\, \xi\rangle, \qquad R_1 |x;\, \rho;\, \xi\rangle = \rho_1 |x;\, \rho;\, \xi\rangle \quad (4.5)$$

The solution of this problem yields the desired *isotopic plane waves*, which have the general form

$$|x;\, \rho;\, \xi\rangle = \xi(\rho)|x;\, \rho\rangle \quad (4.6)$$

Here $\xi(\rho)$ is an arbitrary amplitude, and $|x;\, \rho\rangle$ are *basic spacetime kets* given by

$$|x;\, \rho\rangle = \int d\mu(q) \exp\left[-\frac{i}{\hbar} \Lambda^\mu{}_\nu(\bar{q}^2)(x^\nu - q^\nu)\rho_\mu\right]|q\rangle \quad (4.7)$$

Furthermore, one has the following transformation law for these basic spacetime vectors:

$$U_L(q)|x;\, \rho\rangle = |x';\, \rho\rangle \quad (4.8)$$

where, clearly, $x'^\mu = \Lambda^\mu{}_\nu(q^2)x^\nu + q^\mu$, while the eigenvalues ρ_μ remain invariant. This fact bears no relation to Wigner's 'little group' approach, since equation (4.8) is valid for *all* the representative elements $U_L(q)$ of the group (Wigner, 1939; also see Kim and Wigner, 1990).

Poincaré invariance of ρ_μ holds under the action of the whole group because R_0 and R_1 are invariant operators. Thus, one cannot consider the eigenvalues ρ_μ as playing the role of the *linear 2-momentum vector* of a system. They represent two invariant (i.e., *internal*) properties of the system. We thus briefly refer to ρ_μ as the *isotopic linear 2-momentum*. This feature is peculiar to the present theory. In analogy with the familiar notion of an *isotopic angular momentum*, relativistic quantum kinematics (in 1 + 1 dimensions) affords an *isotopic linear momentum*, which is an invariant quantity characterizing the elementary systems of the model.

Of course, within the left regular representation of $\mathcal{P}^{\uparrow}_{+}(1, 1)$, the components of the *linear 2-momentum vector* p_μ are the eigenvalues of the left generators L_μ. In this manner, one obtains *ordinary plane waves*, of the standard Fourier form, which transform in a well-defined manner under the unitary left-operators of the group. Both kind of plane waves (i.e., isotopic and ordinary plane waves) satisfy the eigenvalue equation of the Casimir operator W [cf. equation (3.1)].

We write $\rho^\mu\rho_\mu = \pm m^2c^2$ without loss of generality, because the invariant momenta admit the whole isotopic 2-momentum space for spectra. In this space, one has four kinds of mass shells. It is useful to denote them as follows:

(a) $m(1)$, when $\rho_0 > 0$ and $-\infty < \rho_1 < +\infty$.
(b) $m(2)$, when $\rho_0 < 0$ and $-\infty < \rho_1 < +\infty$.
(c) $m(3)$, when $-\infty < \rho_0 < +\infty$ and $\rho_1 > 0$.
(d) $m(4)$, when $-\infty < \rho_0 < +\infty$ and $\rho_1 < 0$.

Briefly, we write $\rho \in m(\lambda)$, $\lambda = 1, 2, 3, 4$, to denote the kind of mass shell attached to the isotopic ρ_μ. In the sequel we assume $m \neq 0$, and we take $m > 0$. (The massless case $m = 0$ deserves a special discussion and shall be considered elsewhere.) It is also useful to define the following 'index': $\kappa_\lambda = 1$ for $\lambda = 1, 2$ ('real' ρ-states) and $\kappa_\lambda = -1$ for $\lambda = 3, 4$ ('virtual' ρ-states).

4.3. Extremely Singular Plane Waves

Notwithstanding these features, we can identify the notions of *isotopic proper mass* (as obtained from ρ_μ) and *physical proper mass* (as obtained from p_μ). To strengthen this identification, besides the argument based on W already mentioned, let us recall that $[L_\mu, R_\nu] = 0$, and hence we may solve for the simultaneous eigenvalue problem of these operators. The solution is given by extremely *singular* plane waves $|x; \rho; p\rangle$ which are defined within an arbitrary finite amplitude $\psi = \psi(\rho; p)$, as the reader may convince herself or himself. The transformation law of these plane waves reads $U_L(q)|x; \rho; p\rangle = |x'; \rho; p'\rangle$, where $x'^\mu = \Lambda^\mu{}_\nu(q^2)x^\nu + q^\mu$, $p'_\mu = \Lambda^\nu{}_\mu(\bar{q}^2)p_\nu$, and $\rho'_\mu = \rho_\mu$, as expected. These singular states satisfy the following orthogonal transition amplitudes:

$$\langle x; \rho; p \,|\, x'; \rho'; p' \rangle$$
$$= \exp\left[\frac{i}{\hbar}(x'^{\mu} - x^{\mu})p_{\mu}\right]$$
$$\times\, \delta^{(2)}(\rho' - \rho)\delta^{(2)}(p' - p)\delta(\rho^{\mu}\rho_{\mu} - p^{\mu}p_{\mu})\theta[(\rho_0 + \rho_1)(p_0 + p_1)]$$
$$\times\, \theta[(\rho_0 - \rho_1)(p_0 - p_1)] \tag{4.9}$$

where θ denotes the step function, $\theta(x) = 0$ for $x < 0$ and $\theta(x = 1$ for $x > 0$. Owing to the presence of the step functions in equation (4.9), we see that these transition amplitudes are *identically zero* if ρ_μ and p_μ belong to different *kinds* of mass shell. (Tis is so even if $\rho^{\mu}\rho_{\mu} = p^{\mu}p_{\mu}$ coincide.) So for these states the following *constraint* holds implicitly:

$$\rho^{\mu}\rho_{\mu} = p^{\mu}p_{\mu} = \pm m^2c^2 \tag{4.10}$$

and, furthermore, ρ_μ and p_μ *must* belong to the same kind of mass shell. (This is not to say that $\rho_\mu = p_\mu$, necessarily.)

One gets back the isotopic plane waves (4.6) as superpositions of the singular plane waves in p-space. The wavepacket integrals over p-space actually maintain the identification stated in equation (4.10). We shall not further consider these extremely singular states in this paper. The spacetime vectors given in equations (4.7) (which are defined up to an arbitrary amplitude) are the basic eigenkets for building allowable physical states of elementary systems.

4.4. Isotopic Plane-Wave Propagation Kernel

The transition amplitudes for the basic spacetime vectors (4.7) are given by the following brackets:

$$\langle x; \rho \,|\, x'; \rho' \rangle = \delta^{(2)}(\rho - \rho')K_{(0)}(x - x'; \rho) \tag{4.11}$$

as one obtains from evaluating the corresponding Hurwitz invariant integral. To this end, we set the normalization constant $\mu_0 = (2\pi\hbar)^{-2}$ (see Appendix B), and therefore the kernel $K_{(0)}$ is given by

$$K_{(0)}(x; \rho) = \int_{-1}^{1} dq^2\, \gamma^2(q^2) \exp\left\{\frac{i}{\hbar}\gamma(q^2)[(\rho_0 + q^2\rho_1)x^0 + (\rho_1 + q^2\rho_0)x^1]\right\}$$
$$= \int_0^{\infty} \frac{du}{u} \exp\left\{\frac{i}{2\hbar}[(\rho_0 + \rho_1)(x^0 + x^1)u + (\rho_0 - \rho_1)(x^0 - x^1)u^{-1}]\right\} \tag{4.12}$$

In this way we find the following transition amplitudes:

(I) When $(\rho_0 + \rho_1)(\Delta x^0 + \Delta x^1) < 0$ and $(\rho_0 - \rho_1)(\Delta x^0 - \Delta x^1) < 0$, one has

$$\langle x; \rho | x'; \rho' \rangle = -i\pi \delta^{(2)}(\rho - \rho') H_0^{(2)}[\hbar^{-1}\sqrt{(\rho^\mu \rho_\mu)(\Delta x^\nu \Delta x_\nu)}] \quad (4.13a)$$

(II) Otherwise, one has

$$\langle x; \rho | x'; \rho' \rangle = i\pi \delta^{(2)}(\rho - \rho') H_0^{(1)}[\hbar^{-1}\sqrt{(\rho^\mu \rho_\mu)(\Delta x^\nu \Delta x_\nu)}] \quad (4.13b)$$

where $\Delta x = x' - x$. Here $H_0^{(j)}$ denotes the respective Hankel function ($j = 1, 2$).

Hence we see that, as a consequence of the superselection rules, the rigged Hilbert space $\tilde{\mathcal{H}}(\mathcal{P}_+^\uparrow)$ acquires an incoherent structure, since it becomes diagonalized into a continuous system of invariant Hilbert subspaces $\mathcal{H}_\rho$. Each $\mathcal{H}_\rho$ carries an irreducible representation of $\mathcal{P}_+^\uparrow(1, 1)$, labeled by ρ_μ, and corresponds to a coherent Hilbert subspace in which the superposition principle (for a fixed value of ρ_μ) holds. We also notice the 'regularization' of the theory, as shown by the presence of $\delta^{(2)}(\rho' - \rho)$ in equations (4.13), which stems from the superselection rules.

5. MASSIVE 2-SPINORS: THE DIRAC EQUATION

As we see from equations (4.13), the transition amplitudes $\langle x; \rho | x'; \rho' \rangle$ fail to be identically zero when $x - x'$ is spacelike, even if $\rho^\mu \rho_\mu > 0$. This means that the basic vectors $|x; \rho\rangle$ do not satisfy *microscopic causality.* [In the last analysis, this is due to the presence of the 'cosine' component in the kernel $K_{(0)}(x; \rho)$ defined in equation (4.12).] So we need to find basic spacetime kets which are consistent with the *principle of microscopic* causality.

5.1. The Jordan–Pauli Propagation Kernel

To find such *causal spacetime vectors,* we shall proceed as follows. First, we decompose the kets $|x; \rho\rangle$ defined in equation (4.7) into its 'cosine' and 'sine' components. By this we mean the following definitions:

$$|x; \rho; C\rangle = \frac{1}{2}|x; \rho\rangle + \frac{1}{2}|x; -\rho\rangle$$

$$= \int d\mu(q) \cos\left[\frac{1}{\hbar} \Lambda^\mu{}_\nu(\bar{q}^2)(x^\nu - q^\nu)\rho_\mu\right]|q\rangle \quad (5.1a)$$

$$|x; \rho; S\rangle = \frac{1}{2}|x; \rho\rangle - \frac{1}{2}|x; -\rho\rangle$$

$$= -i \int d\mu(q) \sin\left[\frac{1}{\hbar} \Lambda^\mu{}_\nu(\bar{q}^2)(x^\nu - q^\nu)\rho_\mu\right]|q\rangle \quad (5.1b)$$

As a motivation for introducing these vectors, we note that since the isotopic eigenvalues ρ_μ are invariants, both the 'cosine' and the 'sine' spacetime kets carry a geometric representation:

$$U_L(q)|x; \rho; C\rangle = |x'; \rho; C\rangle, \qquad U_L(q)|x; \rho; S\rangle = |x'; \rho; S\rangle \quad (5.2)$$

Moreover, we easily obtain the transition amplitudes between S-vectors and C-vectors; namely, we get

$$\langle x; \rho; S|x'; \rho'; C\rangle = \frac{i}{2}[\delta^{(2)}(\rho - \rho') - \delta^{(2)}(\rho + \rho')]S_{(0)}(x - x'; \rho) \quad (5.3a)$$

$$\langle x; \rho; C|x'; \rho'; S\rangle = \frac{i}{2}[\delta^{(2)}(\rho - \rho') + \rho^{(2)}(\rho + \rho')]S_{(0)}(x - x'; \rho) \quad (5.3b)$$

where the new propagation kernel $S_{(0)}$ corresponds to

$$\begin{aligned} S_{(0)}(x;\rho) &= \int_{-\infty}^{\infty} dq^3 \sin\left\{\frac{1}{2\hbar}[e^{q^3}(\rho_0+\rho_1)(x^0+x^1) + e^{-q^3}(\rho_0-\rho_1)(x^0-x^1)]\right\} \\ &= \int_0^{\infty} \frac{du}{u} \sin\left\{\frac{1}{2\hbar}[(\rho_0+\rho_1)(x^0+x^1)u + (\rho_0-\rho_1)(x^0-x^1)/u]\right\} \\ &= \pm\pi\theta[(\rho^\mu\rho_\mu)(x^\nu x_\nu)]J_0[(1/\hbar)\sqrt{(\rho^\mu\rho_\mu)(x^\nu x_\nu)}] \end{aligned} \quad (5.4)$$

θ denotes the step function, and J_0 is the Bessel function of the first kind (of order zero). When $\rho^\mu\rho_\mu > 0$, $S_{(0)}(x; \rho)$ is precisely the *Jordan–Pauli causal propagation function* in 1 + 1 dimensions. As is well known, all the propagators used in relativistic quantum theory [including the *noncausal* ones, like $\Delta_1(x)$] can be obtained by means of manipulations performed on the *causal* Jordan–Pauli propagation function, which thus plays the fundamental role. (Within the present quantum kinematic perspective, this matter will be further discussed in a forthcoming paper.)

A detailed discussion in spacetime of the "$\pm\pi$" factor that figures in equation (5.4) in connection with the four kinds of mass shells in ρ-space leads to the following results:

1. For $\lambda = 1$, one gets "$+\pi$" within the future light-cone ($x^0 > 0$), and "$-\pi$" within the past light-cone ($x^0 < 0$).
2. For $\lambda = 2$, one gets "$-\pi$" within the future light-cone ($x^0 > 0$), and "$+\pi$" within the past light-cone ($x^0 < 0$).
3. For $\lambda = 3$ (outside the $\{x\}$ light-cone), one gets "$+\pi$" when $x^1 > 0$, and "$-\pi$" when $x^1 < 0$.
4. For $\lambda = 4$ (outside the $\{x\}$ light-cone), one gets "$+\pi$" when $x^1 < 0$, and "$-\pi$" when $x^1 > 0$.

Of course, in all the remaining cases [that is, when $(\rho^{\mu}\rho_{\mu})(x^{\nu}x_{\nu}) < 0$], one has $S_{(0)} \equiv 0$.

5.2. The Dirac Equation Regained

The following properties of the S and C spacetime kets are immediate:

$$R_0 |x;\, \rho;\, S\rangle = \rho_0 |x;\, \rho;\, C\rangle, \qquad R_1 |x;\, \rho;\, S\rangle = \rho_1 |x;\, \rho;\, C\rangle \tag{5.5a}$$

$$R_0 |x;\, \rho;\, C\rangle = \rho_0 |x;\, \rho;\, S\rangle, \qquad R_1 |x;\, \rho;\, C\rangle = \rho_1 |x;\, \rho;\, S\rangle \tag{5.5b}$$

Using *left-invariant light-cone momentum operators,* $R_+ = R_0 + R_1$ and $R_- = R_0 - R_1$, these can be also written as

$$R_+ |x; \rho; S\rangle = (\rho_0 + \rho_1) |x; \rho; C\rangle, \qquad R_- |x; \rho; C\rangle = (\rho_0 - \rho_1) |x; \rho; S\rangle \tag{5.6a}$$

$$R_+ |x; \rho, C\rangle = (\rho_0 + \rho_1) |x; \rho; S\rangle, \qquad R_- |x; \rho; S\rangle = (\rho_0 - \rho_1) |x; \rho; C\rangle \tag{5.6b}$$

[Both systems of equations in (5.6) are in fact equivalent.] So we see that the microscopic causality condition produces new kinds of spacetime vectors, which obey 'crossed eigenvalue schemes' á la Dirac. In the sequel we adopt the 'crossed scheme' presented in equations (5.6a).

Since the linear momentum operators yield

$$L_{\mu} |x;\, \rho;\, S\rangle = i\hbar(\partial/\partial x^{\mu}) |x;\, \rho;\, S\rangle, \qquad L_{\mu} |x;\, \rho;\, C\rangle = i\hbar(\partial/\partial x^{\mu}) |x;\, \rho;\, C\rangle \tag{5.7}$$

and [recalling equations (3.4a) and (3.4b)] the expressions for the *invariant* light-cone momentum operators are given by

$$R_+ = \gamma(Q^2)(1 - Q^2)(L_0 + L_1), \qquad R_- = \gamma(Q^2)(1 + Q^2)(L_0 - L_1) \tag{5.8}$$

we are ready to build *2-component isotopic spinors* which obey the *Dirac equation*. In fact, equation (5.6a) can be cast in the forms

$$i\hbar[(\partial/\partial x^0) + (\partial/\partial x^1)]a_{\lambda}(\rho)\gamma(Q^2)(1 - Q^2) |x; \rho; S\rangle = mcb_{\lambda}(\rho) |x; \rho; C\rangle \tag{5.9a}$$

$$i\hbar[(\partial/\partial x^0) - (\partial/\partial x^1)]b_{\lambda}(\rho)\gamma(Q^2)(1 + Q^2) |x; \rho; C\rangle = \kappa_{\lambda} mca_{\lambda}(\rho) |x; \rho; S\rangle \tag{5.9b}$$

without loss of generality, where the *amplitudes* $a_{\lambda}(\rho)$ and $b_{\lambda}(\rho)$ are arbitrary c-numbers, provided they satisfy the constraints

$$(\rho_0 + \rho_1)a_{\lambda}(\rho) = mcb_{\lambda}(\rho), \qquad (\rho_0 - \rho_1)b_{\lambda}(\rho) = \kappa_{\lambda} mca_{\lambda}(\rho) \tag{5.10}$$

These equations have nontrivial solutions *on the mass shell* [that is, whenever $\rho^{\mu}\rho_{\mu} = \kappa_{\lambda} m^2 c^2$]. On the other hand, *off the mass shell*, one has $a_{\lambda}(\rho) \equiv 0$ and $b_{\lambda}(\rho) \equiv 0$. At this point, one needs to recall that the isotopic plane waves are defined within an amplitude $\xi(\rho)$ [cf. equation (4.6)]. The same is true, *sensu stricto*, for the S and C spacetime kets defined in equations (5.1). Introducing $|S\rangle \to a_\lambda |S\rangle$ and $|C\rangle \to b_\lambda |C\rangle$ in equations (5.5) leads rather naturally to the 'crossed eigenvalue scheme' for the operators shown in equations (5.9), provided the constraints (5.10) are satisfied. We omit the details of this analysis.

Next, it is rather clear that in order to obtain well-defined wave equations in spacetime we must get rid of the Q^2 in equations (5.9). To this end, we include the action of the boost operator Q^2 in the definition of the basic spacetime vectors themselves. So let us define new basic kets as follows:

$$|x; \rho; (-)\rangle = \left(\frac{1 - Q^2}{1 + Q^2}\right)^{1/4} |x; \rho; S\rangle, \qquad |x; \rho; (+)\rangle = \left(\frac{1 + Q^2}{1 - Q^2}\right)^{1/4} |x; \rho; C\rangle \tag{5.11}$$

In this fashion, we cast equations (5.9) in the typical form of two coupled first-order partial differential equations, i.e., we obtain

$$i\hbar[(\partial/\partial x^0) + (\partial/\partial x^1)]a_\lambda(\rho)|x; \rho; (-)\rangle = mcb_\lambda(\rho)|x; \rho; (+)\rangle \tag{5.12a}$$

$$i\hbar[(\partial/\partial x^0) - (\partial/\partial x^1)]b_\lambda(\rho)|x; \rho; (+)\rangle = \kappa_\lambda mca_\lambda(\rho)|x; \rho; (-)\rangle \tag{5.12b}$$

These correspond to the *Dirac equation* in 1 + 1 dimensions indeed (Rosen, 1969), in which $mc > 0$ now plays the role of an *eigenvalue*. We wish to remark that equations (5.12) do not just arise as the "most simple" (or the "most elegant") way out of the "Q^2 impasse" manifested by equations (5.9). A more detailed discussion shows that they actually arise as the *unique* way out of that problem. In other words, equations (5.11) represent a *sufficient and necessary construct* of the theory.

Furthermore, given their definition in (5.11) [and recalling equations (2.12)], the transformation law for these spacetime kets, under the action of the unitary operators of $\mathcal{P}_+^\uparrow(1, 1)$, reads

$$U_L(q)|x; \rho; (-)\rangle = \left(\frac{1 + q^2}{1 - q^2}\right)^{1/4} |x'; \rho; (-)\rangle \tag{5.13a}$$

$$U_L(q)|x; \rho; (+)\rangle = \left(\frac{1 - q^2}{1 + q^2}\right)^{1/4} |x'; \rho; (+)\rangle \tag{5.13b}$$

instead of equation (4.1). One recognizes these as precisely the transformation law obeyed by the components of Dirac '2-spinors' in 2-dimensional spacetime (Rosen, 1969). We also wish to underline here that, *by construction*, these components of 2-spinor-vectors are general solutions to the *Dirac equations* (5.12), for this means that these wave equations have been *deduced* within the quantum kinematic theory, together with the causality demand of the theory of special relativity.

5.3. Some Features of Two-Fermion Transition Amplitudes

Let us finally examine the new basic transition amplitudes $\langle x'; \rho'; (\pm) | x; \rho; (\pm)\rangle$ more closely. Certainly,

$$\langle x; \rho; (-) | x'; \rho'; (+)\rangle = \langle x; \rho; S | x'; \rho'; C\rangle \tag{5.14a}$$

$$\langle x; \rho; (+) | x'; \rho'; (-)\rangle = \langle x; \rho; C | x'; \rho'; S\rangle \tag{5.14b}$$

still hold, as given in equations (5.3). We then concentrate our attention on the brackets:

$$\langle x; \rho; (-) | x'; \rho; (-)\rangle = \frac{i}{2}[\delta^{(2)}(\rho - \rho') - \delta^{(2)}(\rho + \rho')]S_{(-)}(x - x'; \rho) \tag{5.15a}$$

$$\langle x; \rho; (+) | x'; \rho'; (+)\rangle = \frac{i}{2}[\delta^{(2)}(\rho - \rho') + \delta^{(2)}(\rho + \rho')]S_{(+)}(x - x'; \rho) \tag{5.15b}$$

whose kernels are given by

$$S_{(\pm)}(x; \rho) = -i \int_{-\infty}^{\infty} dq^3\, e^{\pm q^3} \cos\left\{\frac{1}{2\hbar}\,[e^{q^3}(\rho_0 + \rho_1)(x^0 + x^1) + e^{-q^3}(\rho_0 - \rho_1)(x^0 - x^1)]\right\} \tag{5.16}$$

These lead to the familiar result:

$$S_{(-)}(x; \rho) = -i\hbar\, \frac{\rho_0 + \rho_1}{\rho^\mu \rho_\mu}\,[(\partial/\partial x^0) - (\partial/\partial x^2)]S_{(0)}(x; \rho) \tag{5.17a}$$

$$S_{(+)}(x; \rho) = -i\hbar\, \frac{\rho_0 - \rho_1}{\rho^\mu \rho_\mu}\,[(\partial/\partial x^0) + (\partial/\partial x^1)]S_{(0)}(x; \rho) \tag{5.17b}$$

where $S_{(0)}(x; \rho)$ has been defined in equation (5.4). From equation (5.4) the well-known properties of $S_{(0)}$ follow immediately:

$$\rho^\mu \rho_\mu > 0 \Rightarrow \lim_{x^0 \to 0} S_{(0)}(x; \rho) = 0, \qquad \rho^\mu \rho_\mu < 0 \Rightarrow \lim_{x^1 \to 0} S_{(0)}(x; \rho) = 0 \tag{5.18}$$

On the other hand, it can be shown after a few steps that

$$S_{(\pm)}(x; \rho) = S_{(\pm)}(-x; \rho) = S_{(\pm)}(x; -\rho) \tag{5.19}$$

In fact, these kernels can be also written as follows:

$$S_{(-)}(x; \rho) = -i \int_0^{\infty} \frac{du}{u^2} \cos\left\{\frac{1}{2\hbar}\left[(\rho_0 + \rho_1)(x^0 + x^1)u + \frac{(\rho_0 - \rho_1)(x^0 - x^1)}{u}\right]\right\} \quad (5.19a)$$

$$S_{(+)}(x; \rho) = -i \int_0^{\infty} du \cos\left\{\frac{1}{2\hbar}\left[(\rho_0 + \rho_1)(x^0 + x^1)u + \frac{(\rho_0 - \rho_1)(x^0 - x^1)}{u}\right]\right\} \quad (5.19b)$$

wherefrom a straightforward calculation yields another well-known basic result:

$$\lim_{z^0 \to 0} S_{(\pm)}(x; \rho) = -\frac{2\pi i\hbar}{\rho_0 + \rho_1} \delta(x^1) \quad (5.20)$$

In this manner, quantum kinematics produces all the mathematical ingredients needed for developing a *2-dimensional isotopic-spinor* formalism describing the quantum kinematic theory of *free fermions* in a 2-dimensional flat spacetime. This theory will be examined in another paper.

6. CONCLUDING REMARKS AND PERSPECTIVES

Heuristic interest is attached to the Dirac equation in 1 + 1 dimensions (Rosen, 1969). The fact that with non-Abelian quantum kinematics of the Poincaré group $\mathcal{P}_+^{\uparrow}(1, 1)$ one is able to *deduce* the Dirac equation, and the Jordan–Pauli propagation kernel, is very reassuring for the general purposes of this new quantization scheme. Although the present analysis does not bear on dynamical questions (as, for instance, in Krause, 1986, 1996), in order to understand the real significance of this paper, it is very important to bear in mind that here we have followed the *general deductive procedure* of quantum kinematics, as developed in our previous work (Krause, 1994a, 1997a).

In fancy language, it could be said that this paper presents a kind of group-theoretic "radiography" of the kinematic structure of the free-particle Dirac equation and its propagation kernel in 1 + 1 dimensions. Indeed, a kind of self-contained origin of the structure of 2-spinors has come to the fore as a necessary construct of the formalism. These well-known structures become illuminated from a new perspective, which includes the most fundamental requirements of special relativity and quantum theory, melded into just *one* consistent and general group-theoretic picture. [Certainly, "there is a pleasure in recognizing old things from a new point of view" (Feynman).]

As a matter of fact, as we have already mentioned in the Introduction, quantum kinematics affords a new group-theoretic method for obtaining the propagation kernel of a system. The reader may have noticed that all the propagators calculated in this paper have been obtained by evaluating some Hurwitz invariant integrals over the group manifold, which correspond to transition probability amplitudes between allowable configuration states obtained from the superselection rules. There may be problems (not touched on in this paper) for which this new formulation can offer some distinctive advantages.

The usefulness of quantum kinematic theory does not stop simply at providing another method for deducing wave equations and their propagation kernels. Quantum kinematics of the Poincaré group in 1 + 1 dimensions may afford interesting *toy models* of relativistic elementary quantum systems. For instance, based on the introductory theory developed in this paper, one could further study the *Lorentz-invariant* canonical commutation relations $[\bar{Q}^\mu, R_\nu] = -i\hbar\delta^\mu_\nu$, presented in equation (2.9), in order to obtain (say) a *Lorentz-invariant SU*(2) *algebra* (in terms of $\hat{a}_\mu$ and $\hat{a}^\dagger_\mu$ Lorentz-invariant ladder operators), and then use $|\psi\rangle = |JM\rangle \in \mathcal{H}(\mathcal{P}^\uparrow_+)$ (with $J = 0, 1/2, 1, 3/2, \ldots$; $M = -J, -J + 1, \ldots, J - 1, J$) to produce the two wave-function components of 2-spinors. In fact, there is room to choose $|\psi\rangle$ in a physically interesting way, in order to obtain *isotopic multiplets* of 2-spinor states, and then try to interpret them reasonably. Another instance of an *intrinsic Lorentz-invariant algebra* that appears within the quantum kinematic theory of $\mathcal{P}^\uparrow_+(1, 1)$ is discussed in Krause (1993b). This is the algebra of the group *SU*(1, 1), often used in particle physics, which also appears by itself as an *internal symmetry* of Lorentz-invariant systems. Much remains to be done on this most intriguing subject.

Isotopic 2-momentum ρ_μ is another mechanical curiosity of this theory, whose actual physical meaning remains to be understood. Let us here only add that if one quantizes the universal covering group of the full Poincaré group in 4-dimensional spacetime, one implicitly quantizes the *external SU*(2) symmetry group. Hence, owing to typical quantum kinematic features, this brings an *isotopic SU*(2) algebra onto the scene (Krause, 1997b). Therefore, besides the 4-dimensional *isotopic linear momentum,* an *isotopic angular momentum* [producing *internal SU*(2) *multiplets*] must arise in the 4-dimensional theory.

Also, the isotopic massless case and the meaning of 'virtual' 2-spinor states, as well as the quantum kinematic theory of the Klein–Gordon equation in 1 + 1 dimensions, are subjects which deserve special study. The present 2-dimensional theory can be extended in several ways indeed. It is quite clear that one must not expect to answer all the pertinent questions (posed by these toy models) by just ne stroke of luck. Rather, long and hard step-by-step work will be needed to this end.

Nevertheless, there are several good reasons to expect that a complete and unambiguous non-Abelian quantum theory is possible to achieve, as a direct group-theoretic generalization of the present quantum formalism, without changing the heuristic rules of physical interpretation. Such an achievement could be a helpful theoretical tool in the realm of elementary particle physics.

APPENDIX A. SOME FEATURES OF POINCARÉ TRANSFORMATIONS

Here we append some features of Poincaré transformations in 2-dimensional spacetime. We develop this issue rather sketchily. The formulas presented in this appendix are quite familiar (most are elementary). However, all of them are employed somewhere in this article. To avoid repetitions, it seems advisable to have them at hand, as references for reading the paper.

We consider the transformation of variables $x^\mu = (x^0, x^1) \to x'^\mu = (x'^0, x'^1)$, given by $x'^\mu = \Lambda^\mu{}_\nu(q^2)x^\nu + q^\mu$; i.e.,

$$\begin{aligned} x'^0 &= \gamma(q^2)(x^0 - q^2 x^1) + q^0 \\ x'^1 &= \gamma(q^2)(x^1 - q^2 x^0) + q^1 \end{aligned} \tag{A.1}$$

The q's are the parameters of the group. Here q^μ, $\mu = 0, 1$, are rigid displacements of the inertial Cartesian frame, and q^2 is the Lorentz boost parameter, and one defines $\gamma(q^2) = [1 - (q^2)^2]^{-1/2}$. The $\Lambda^\mu{}_\nu(q^2)$ denotes a 2×2 proper orthochronous Lorentz matrix. In the present parametrization, the group manifold is given by

$$M(\mathcal{P}^\uparrow_+) = \{-\infty < q^0 < +\infty, -\infty < q^1 < +\infty, -1 < q^2 < +1\} \tag{A.2}$$

and the identity point is $e = (0, 0, 0) \in M(\mathcal{P}^\uparrow_+)$. The *group law* reads

$$\begin{aligned} q''^0 &= g^0(q'; q) = q'^0 + \gamma(q'^2)(q^0 - q'^2 q^1) \\ q''^1 &= g^1(q'; q) = q'^1 + \gamma(q'^2)(q^1 - q'^2 q^0) \\ q''^2 &= g^2(q'; q) = (q'^2 + q^2)(1 + q'^2 q^2)^{-1} \end{aligned} \tag{A.3}$$

and therefore the *group inversion rule* for these parameters follows:

$$\bar{q}^0 = -\gamma(q^2)(q^0 + q^2 q^1), \qquad \bar{q}^1 = -\gamma(q^2)(q^1 + q^2 q^0), \qquad \bar{q}^2 = -q^2 \tag{A.4}$$

The q's are real essential parameters and $M(\mathcal{P}^\uparrow_+)$ is a noncompact, connected and simply connected, 3-dimensional space.

One defines (*right* and *left*) *transport matrices* in the group manifold, which are given by $R^b_a(q) = \lim_{q' \to e} \partial'_a g^b(q'; q)$ and $L^b_a(q) = \lim_{q' \to e} \partial'_a g^b(q; q')$. Hence, one gets the following transport matrices in $M(\mathcal{P}^\uparrow_+)$:

$$R_a^b(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q^1 & -q^0 & \gamma^{-2} \end{bmatrix}, \qquad L_a^b(q) = \begin{bmatrix} \gamma & -\gamma q^2 & 0 \\ -\gamma q^2 & \gamma & 0 \\ 0 & 0 & \gamma^{-2} \end{bmatrix} \tag{A.5}$$

(In these matrices "a" labels the rows and "b" labels the columns.) In quantum kinematics one also needs the corresponding *inverse transport matrices*, which one defines as follows: $\bar{R}_a^b(q) = \lim_{q'\to q} \partial'_a g^b(q'; \bar{q})$ and $\bar{L}_a^b(q) = \lim_{q'\to q} \partial'_a g^b(\bar{q}; q^1)$. Thus

$$\bar{R}_a^b(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma^2 q^1 & \gamma^2 q^0 & \gamma^2 \end{bmatrix}, \qquad \bar{L}_a^b(q) = \begin{bmatrix} \gamma & \gamma q^2 & 0 \\ \gamma q^2 & \gamma & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix} \tag{A.6}$$

So one immediately obtains the *adjoint representation*, which is carried by the matrices $A_a^b(q) = R_a^c(q)\bar{L}_c^b(q)$, i.e.,

$$A_a^b(q) = \begin{bmatrix} \gamma & \gamma q^2 & 0 \\ \gamma q^2 & \gamma & 0 \\ \bar{q}^1 & \bar{q}^0 & 1 \end{bmatrix} \tag{A.7}$$

from which one gets $A(q) = \det[A_a^b(q)] = R(q)\bar{L}(q) = 1$. Hence, $\mathcal{P}_+^\uparrow(1, 1)$ is *unimodular.* All these matrices play important roles in quantum kinematics.

One next defines Lie (*right* and *left*) vector fields on $M(\mathcal{P}_+^\uparrow)$: $X_a(q) = R_a^b(q)\partial_b$, $Y_a(q) = L_a^b(q)\partial_b$. So one obtains the operators

$$X_0 = \partial_0, \qquad X_1 = \partial_1, \qquad X_2 = -q^1\partial_0 - q^0\partial_1 + \gamma^{-2}\partial_2 \tag{A.8a}$$

$$Y_0 = \gamma(\partial_0 - q^2\partial_1), \qquad Y_1 = \gamma(\partial_1 - q^2\partial_2), \qquad Y_2 = \gamma^{-2}\partial_2 \tag{A.8b}$$

These satisfy the well known Lie algebra:

$$[X_0, X_1] = 0, \qquad [X_0, X_2] = -X_1, \qquad [X_1, X_2] = -X_0 \tag{A.9a}$$

$$[Y_0, Y_1] = 0, \qquad [Y_0, Y_2] = Y_1, \qquad [Y_1, Y_2] = Y_0 \tag{A.9b}$$

$$[X_a, Y_b] = 0, \qquad a, b = 0, 1, 2 \tag{A.9c}$$

Hence, the nonzero structure constants are $f_{20}^1 = f_{21}^0 = 1$. This ends the required formulary.

APPENDIX B. THE REGULAR REPRESENTATION OF $\mathcal{P}_+^\uparrow$ REVISITED

This appendix briefly reviews the formalism of the regular representation, since this formalism contains the core of quantum kinematic theory.

The regular representation of Lie groups is a well-known subject that has been amply studied by mathematicians (Naimark and Stern, 1982). The only aim here (besides introducing some further notation) is to present a *unified* formalism for the simultaneous description of both (the *left* and the *right*) regular representations of $\mathcal{P}_+^{\uparrow}(1, 1)$. To this end, we shall stick to the standard Dirac notation used in quantum mechanics (i.e., in terms of "kets" and "bras"). This appendix is also a useful formulary at hand for reading this paper.

Let us then recall some features of both regular representations of $\mathcal{P}_+^{\uparrow}(1, 1)$. Since the group is unimodular [i.e., $R(q) = L(q) = \gamma^{-2}$ (cf. Appendix A)], we define the Hurwitz measure in $M(\mathcal{P}_+^{\uparrow})$:

$$d\mu(q) = \mu_0\gamma^2(q^2)\, dq^0\, dq^1\, dq^2 \tag{B.1}$$

where μ_0 is a normalization constant. [In this paper we set $\mu_0 = (2\pi\hbar)^{-2}$.] This measure is left- and right-invariant under the group law (A.3). We then introduce the Hilbert space $\mathcal{H}(\mathcal{P}_+^{\uparrow})$ that carries the two regular representations of the Poincaré group in 1 + 1 dimensions. As is well known, this space is given by the set of all complex functions $\psi(q) = \psi(q^0, q^1, q^2)$, defined on the group manifold, which have a finite invariant norm:

$$\langle\psi|\psi\rangle = \int d\mu(q)\, |\psi(q)|^2 < \infty \tag{B.2}$$

Thus, we better introduce the *rigged* Hilbert space $\tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$ to handle this subject. This space is endowed with a continuous complete orthogonal basis $\{|q\rangle = |q^0, q^1, q^2\rangle\}$ which is consistent with the invariant measure (B.1); namely, one has

$$\mu_0 \iint dq^0\, dq^1 \int_{-1}^{1} dq^2\, \gamma^2(q^2)|q^0, q^1, q^2\rangle\langle q^0, q^1, q^2| = I \tag{B.3}$$

$$\langle q'^0, q'^1, q'^2|q^0, q^1, q^2\rangle = \mu_0^{-1}\gamma^{-2}(q^2)\delta(q'^0 - q^0)(q'^1 - q^1)\delta(q'^2 - q^2) \tag{B.4}$$

where μ_0 denotes an arbitrary constant of normalization. There is a one-to-one correspondence indeed: $|q\rangle \leftrightarrow q \in M(\mathcal{P}_+^{\uparrow})$, and therefore for any given vector $|\psi\rangle \in \mathcal{H}(\mathcal{P}_+^{\uparrow})$ one defines the wave function $\psi(q) = \langle q|\psi\rangle$, which satisfies (B.2). Conversely, given any function $\psi(q)$ satisfying (B.2), one defines the vector

$$|\psi\rangle = \int d\mu(q)\, \psi(q)|q\rangle \in \mathcal{H}(\mathcal{P}_+^{\uparrow}) \tag{B.5}$$

as one does in ordinary quantum mechanics (that is, when one considers Heisenberg's *Abelian* quantum kinematics of rigid Cartesian translations). In this sense, the present construct is rather simple and looks quite familiar.

However, it produces interesting novelties, wing mainly to the *non-Abelian* structure of $\mathscr{P}_+^\uparrow(1, 1)$.

We next introduce the representative operators of $\mathscr{P}_+^\uparrow(1, 1)$ in $\tilde{\mathscr{H}}(\mathscr{P}_+^\uparrow)$, in the following fashion:

$$U_L(q) = \int d\mu(q') \,|g(q; q')\rangle\langle q'|, \qquad U_R(q) = \int d\mu(q')\,|g(q'; q)\rangle\langle q'| \tag{B.6}$$

where the group multiplication functions $g^a(q'q)$ are given in equations (A.3). These operators satisfy the group property,

$$U_L(q')U_L(q) = U_L[g(q'; q)], \qquad U_R(q')U_R(q) = U_R[g(q; q')] \tag{B.7}$$

they are unitary,

$$U_L^\dagger(q) = U_L(\bar{q}) = U_L^{-1}(q), \qquad U_R^\dagger(q) = U_R(\bar{q}) = U_R^{-1}(q) \tag{B.8}$$

[cf. equations (A.4)], and moreover, left- and right-operators commute:

$$U_L(q')U_R(q) = U_R(q)U_L(q') \tag{B.9}$$

for all q', $q \in M$. Of course, they yield

$$U_L(q)|q'\rangle = |g(q; q')\rangle, \qquad U_R(q)|q'\rangle = |g(q'; q)\rangle \tag{B.10}$$

and therefore if one defines $\mathscr{P}_+^\uparrow$-transformed vectors in $\mathscr{H}(\mathscr{P}_+^\uparrow)$, namely

$$U_L(q)|\psi\rangle = |\psi_q^{(L)}\rangle, \qquad U_R(q)|\psi\rangle = |\psi_q^{(R)}\rangle \tag{B.11}$$

one obtains the corresponding transformation laws for wave functions $\psi(q)$ defined on the group manifold. These read [cf. also equation (4.2)]:

$$\psi_q^{(L)}(q') = \psi[g(\bar{q}; q')], \qquad \psi_q^{(R)}(q') = \psi[g(q'; \bar{q})] \tag{B.12}$$

These formulas show neatly that we are handling the regular representations of the group, for these are the basic definitions used in the current treatment of this subject by mathematicians (Naimark and Stern, 1982). (We deem our treatment as much simpler.)

The generators are defined in the neighborhood of the identity, say

$$U_L(\delta q) = I - (i/\hbar)\delta q^a L_a, \qquad U_R(\delta q) = I - (i/\hbar)\delta q^a R_a \tag{B.13}$$

The kinematic of the generators is well known. They transform as vectors of the adjoint representation:

$$U_L^\dagger(q)L_aU_L(q) = A_a^b(q)L_b, \qquad U_R^\dagger(q)R_aU_R(q) = A_a^b(\bar{q})R_b \tag{B.14}$$

which stem from equations (B.7). Of course, the Lie algebra obeyed by the generators is an immediate consequence of these results; thus we get [cf. equations (A.9)]

$$[L_0, L_1] = 0, \qquad [L_0, L_2] = i\hbar L_1, \qquad [L_1, L_2] = i\hbar L_0 \quad \text{(B.15a)}$$

$$[R_0, R_1] = 0, \qquad [R_0, R_2] = -i\hbar R_1, \qquad [R_1, R_2] = -i\hbar R_0 \quad \text{(B.15b)}$$

Furthermore, equations (B.10) yield the following realizations of the generators when acting on the basic kets $|q\rangle$ of $\tilde{\mathcal{H}}(\mathcal{P}_+^{\uparrow})$:

$$L_a|q\rangle = i\hbar X_a(q)|q\rangle, \qquad R_a|q\rangle = i\hbar Y_a(q)|q\rangle \qquad \text{(B.16)}$$

Hence, using equations (A.8), we obtain explicitly

$$\begin{aligned}\langle q|L_0|\psi\rangle &= -i\hbar\partial_0\psi(q)\\ \langle q|L_1|\psi\rangle &= -i\hbar\partial_1\psi(q)\\ \langle q|L_2|\psi\rangle &= i\hbar(q^0\partial_1 + q^1\partial_0 - \gamma^{-2}\partial_2)\psi(q)\end{aligned} \qquad \text{(B.17a)}$$

and

$$\begin{aligned}\langle q|R_0|\psi\rangle &= -i\hbar\gamma(\partial_0 - q^2\partial_1)\psi(q)\\ \langle q|R_1|\psi\rangle &= -i\hbar\gamma(\partial_1 - q^2\partial_0)\psi(q)\\ \langle q|R_2|\psi\rangle &= -i\hbar\gamma^{-2}\partial_2\psi(q)\end{aligned} \qquad \text{(B.17b)}$$

for all $|\psi\rangle \in \mathcal{H}(\mathcal{P}_+^{\uparrow})$. Finally, note that equation (B.9) means

$$U_L^{\dagger}(q)R_aU_L(q) = R_a, \qquad U_R^{\dagger}(q)L_aU_R(q) = L_a \qquad \text{(B.18)}$$

$a = 0, 1, 2$, from which

$$[L_a, R_b] = 0 \qquad (a, b = 0, 1, 2) \qquad \text{(B.19)}$$

follows. That much of the regular representation of $\mathcal{P}_+^{\dagger}(1, 1)$ is needed in this paper.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Arno Bohm for his friendly hospitality at the University of Texas at Austin, where much of this work was done.

The material presented in this paper is based upon work supported by FONDECYT, grant No. 1960256.

REFERENCES

Bargmann, V., and Wigner, E. P. (1946). *Proceedings of the National Academy of Science of the USA,* **34**, 211.

Biedenharn, L. C., and Lohe, M. A. (1995). *Quantum Group Symmetry and q-Tensor Algebras,* World Scientific, Singapore.

Bose, S. K. (1996). *Journal of Mathematical Physics,* **37**, 2376.

Fonda, L., and Ghirardi, G. C. (1970). *Symmetry Principles in Quantum Physics,* Marcel Dekker, New York.

Jammer, M. (1989). *The Conceptual Development of Quantum Mechanics,* AIP, Thomas Publishers, New York.

Kim, Y. S., and Wigner, E. P. (1990). *Journal of Mathematical Physics,* **31**, 55.

Komar, A. (1971). Semantic foundations of the quantization program, in *Problems in the Foundations of Physics,* M. Bunge, ed., Springer-Verlag, Berlin.

Krause, J. (1985). *Journal of Physics A,* **18**, 1309.

Krause, J. (1986). *Journal of Mathematical Physics,* **27**, 2922.

Krause, J. (1988). *Journal of Mathematical Physics,* **29**, 393.

Krause, J. (1991). *Journal of Mathematical Physics,* **32**, 348.

Krause, J. (1993a). *International Journal of Theoretical Physics,* **32**, 1363.

Krause, J. (1993b). *Journal of Physics A,* **26**, 6285.

Krause, J. (1994a). *International Journal of Theoretical Physics,* **33**, 1617.

Krause, J. (1994b). *Journal of Mathematical Physics,* **35**, 5734.

Krause, J. (1996). *Physical Review A,* **54**, 4691.

Krause, J. (1997a). *International Journal of Theoretical Physics,* **36**, 841.

Krause, J. (1997b). $SU(2)$ quantum kinematics: Rotation-observable vs angular-momentum generalized commutation relations, submitted for publication.

Naimark, M. A., and Stern, A. (1982). *Theory of Group Representations,* Springer, Berlin.

Pillin, M. (1994). *Journal of Mathematical Physics,* **35**, 2804.

Rosen, G. (1969). *Formulations of Classical and Quantum Dynamical Theory,* Academic Press, New York.

Weyl, H. (1931). *The Theory of Groups and Quantum Mechanics,* Dover, New York.

Wigner, E. P. (1939). *Annals of Mathematics,* **149**, 40.